Examples of Vector Spaces

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Example 1

Let F be any field and V be the set of all n–tuples:

$$V = \{(x_1, x_2, \dots, x_n) : \; x_1, x_2, \dots, x_n \in F\}$$

then V is denoted by F^n. Vector addition and scalar multiplication for F^n is defined as below $\forall x, y \in F^n$ and $c \in F$, where $x_i, y_i \in F$ and $i = 1, 2, \dots, n$:

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ c \cdot x &= (c \cdot x_1, c \cdot x_2, \dots, c \cdot x_n) \end{aligned}$$

Verify that the $F^n(F)$ forms a vector space.

Solution:

Vector addition for $F^n(F)$ is defined as,

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

where $x_i, y_i \in F$, and since addition of elements of a field is closed under that field, $x_i + y_i \in F$, and so, $(x + y) \in F^n$, $\forall x, y \in F^n$. Therefore, closure property is satisfied for vector addition.

$$\begin{aligned} x + y = (x_1 + y_1, \dots, x_n + y_n) = (y_1 + x_1, \dots, y_n + x_n) = y + x && \text{(Commutativity)} \\ x + (y + z) = (x_1 + (y_1 + z_1), \dots) = ((x_1 + y_1) + z_1), \dots) = (x + y) + z && \text{(Associativity)} \\ x + 0 = (x_1 + 0, \dots, x_n + 0) = x && \text{(Additive Identity)} \\ x + (-x) = (x_1 + (-x_1), \dots, x_n + (-x_n)) = 0 && \text{(Additive Inverse)} \end{aligned}$$

Scalar multiplication for $F^n(F)$ is defined as,

$$c \cdot x = (c \cdot x_1, c \cdot x_2, \dots, c \cdot x_n)$$

where $c, x_i \in F$, and since multiplication of elements of a field is closed under that field, $c \cdot x_i \in F$, and so, $c \cdot x \in F^n$, $\forall x \in F^n$ and $c \in F$. Therefore, closure property is satisfied for scalar multiplication. $\forall \alpha, \beta \in F$:

$$\begin{aligned} 1 \cdot x = (1 \cdot x_1, 1 \cdot x_2, \dots, 1 \cdot x_n) = x && \text{(Multiplicative Identity)} \\ (\alpha\beta)x = ((\alpha\beta)x_1, \dots, (\alpha\beta)x_n) = (\alpha(\beta x_1), \dots, \alpha(\beta x_n)) = \alpha(\beta x) && \text{(Associativity)} \\ \alpha \cdot (x + y) = (\alpha \cdot (x_1 + y_1), \dots) = ((\alpha \cdot x_1) + (\alpha \cdot y_1), \dots) = (\alpha \cdot x) + (\alpha \cdot y) && \text{(Distributivity)} \\ (\alpha + \beta) \cdot x = ((\alpha + \beta) \cdot x_1, \dots) = (\alpha \cdot x_1 + \beta \cdot x_1, \dots) = (\alpha \cdot x) + (\beta \cdot x) && \text{(Distributivity)} \end{aligned}$$

Hence, $F^n(F)$ forms a vector space.

Example 2

Verify that the set of all matrices of order $m \times n$, having elements from field F, denoted by $F^{m\times n}(F)$, forms a vector space.

Solution:

Let F be any field, $m, n \in \mathbb{Z}$, and $F^{m\times n}$ be the set of all $m \times n$ matrices over F.

$$F^{m\times n} = \{(a_{ij})_{m\times n} \mid a_{ij} \in F\}$$

Vector addition is defined as follows $\forall A, B \in F^{m\times n}$:

$$A + B = (a_{ij} + b_{ij})_{m\times n}$$
$$\forall i = 1, \dots, n,\ \forall j = 1, \dots, n$$

where,

$$A = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$$B = (b_{ij})_{m\times n} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

Scalar multiplication is defined as $c \cdot A = (c \cdot a_{ij})_{m\times n}$, $\forall c, \alpha, \beta \in F$.

$$\begin{aligned}
A + B &= (a_{ij} + b_{ij})_{m\times n} \quad ((a_{ij} + b_{ij}) \in F \implies \text{Closure Property}) \\
A + B = (a_{ij} + b_{ij})_{m\times n} &= (b_{ij} + a_{ij})_{m\times n} = B + A \quad \text{(Commutativity)} \\
A + (B + C) = (a_{ij} + (b_{ij} + c_{ij}))_{m\times n} &= ((a_{ij} + b_{ij}) + c_{ij})_{m\times n} = (A + B) + C \quad \text{(Associativity)} \\
A + (0)_{m\times n} = (a_{ij} + 0)_{m\times n} &= (a_{ij})_{m\times n} = A \quad \text{(Additive Identity)} \\
A + (-A) = (a_{ij} + (-a_{ij}))_{m\times n} &= (0)_{m\times n} = 0 \quad \text{(Additive Inverse)} \\
c \cdot A &= (c \cdot a_{ij})_{m\times n} \quad ((c \cdot a_{ij}) \in F \implies \text{Closure Property}) \\
(\alpha\beta) \cdot A = ((\alpha\beta) \cdot a_{ij})_{m\times n} &= (\alpha \cdot (\beta a_{ij}))_{m\times n} = \alpha \cdot (\beta A) \quad \text{(Associativity)} \\
(\alpha + \beta) \cdot A &= ((\alpha + \beta) \cdot a_{ij})_{m\times n} \\
&= ((\alpha \cdot a_{ij}) + (\beta \cdot a_{ij}))_{m\times n} \\
&= (\alpha \cdot A) + (\beta \cdot A) \quad \text{(Distributivity)} \\
\alpha \cdot (A + B) &= (\alpha \cdot (a_{ij} + b_{ij}))_{m\times n} \\
&= ((\alpha \cdot a_{ij}) + (\alpha \cdot b_{ij}))_{m\times n} \\
&= (\alpha \cdot A) + (\alpha \cdot B) \quad \text{(Distributivity)}
\end{aligned}$$

Hence, $F^{m\times n}(F)$ forms a vector space.

Example 3

Let S be any non-empty set and F be any field, and let $\mathcal{F}(S, F)$ denote the set of all functions from $S \to F$:

$$\mathcal{F}(S, F) = \{f \mid f : S \to F\}$$

Vector addition is defined as follows, let $f, g \in \mathcal{F}(S, F)$:

$$(f + g)(s) = f(s) + g(s) \qquad (\forall s \in S)$$

and scalar multiplication is defined as follows, let $c \in F$:

$$(c \cdot f)(s) = c \cdot f(s)$$

Solution:

Since $f : S \to F$ and $g : S \to F \implies f(s) + g(s) \in F,\ \forall s \in S$, and so, $(f + g)(s) \in \mathcal{F}(S, F)$. Therefore, closure property for vector addition is satisfied.

$$(f + g)(s) = f(s) + g(s) = g(s) + f(s) = (g + f)(s) \qquad \text{(Commutativity)}$$
$$(f + (g + h))(s) = f(s) + (g(s) + h(s)) = (f(s) + g(s)) + h(s) = ((f + g) + h)(s) \qquad \text{(Associativity)}$$
$$0(s) = 0,\ \forall s \in S \qquad \text{(Additive Identity)}$$
$$(f + (-f))(s) = f(s) + (-f(s)) = 0 \qquad \text{(Additive Inverse)}$$

$(c \cdot f)(s) = c \cdot f(s)$, where $c, f(s) \in F \implies c \cdot f(s) \in F \implies (c \cdot f)(s) \in \mathcal{F}(S, F)$. Therefore, closure property is satisfied for scalar multiplication.

$$f(s) = 1,\ \forall s \in S \qquad \text{(Multiplicative Identity)}$$
$$(\alpha\beta \cdot f)(s) = (\alpha\beta) \cdot f(s) = \alpha \cdot (\beta f(s)) = ((\beta f(s)) \cdot \alpha)(s),\ \forall s \in S \qquad \text{(Associativity)}$$